Lifespan of Trailing Vortices in a Turbulent Atmosphere

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The lifespan of aircraft trailing vortices is controlled by a mutual induction instability excited by atmospheric turbulence. The instability itself is well understood. The purpose here is to incorporate the effects of turbulence and thereby predict wake lifespan as a function of meteorological conditions. Eddies of the relevant size are assumed to lie in the Kolmogorov inertial subrange, characterized by an energy dissipation rate ϵ. The appropriate dimensionless measure of turbulence intensity proves to be $\eta=(\epsilon b)^{1/3}/(\Gamma/2\pi b)$, where Γ is the circulation around the vortices, and b is their separation. Similarity considerations imply a mean wake lifespan of the form $(2\pi b^2/\Gamma)\tau(\eta)$, where τ is a universal function of η. A statistical definition of lifespan is proposed, and τ is computed in the limit of large η, when the vortices are too weak to influence their own deformation. Vortex induction then is included, and τ is computed in the opposite limit of small η by the method of stationary phase. In that limit, vertical currents are the chief excitors of instability. The two asymptotic forms for $\tau(\eta)$ join in a smooth curve in reasonable agreement with the few published data. The paper concludes with a review of a practical method for actively exciting the mutual induction instability. The method would shorten the typical lifespan of a 747 wake by a factor of 3.

I. Introduction

THE vortex wakes of large jet aircraft are hazardous to light planes and impose a constraint on airport operations. Research on mechanisms limiting the lifespan of aircraft wakes consequently has been vigorous, and the fundamentals of one decay mechanism are now well established. An aircraft wing leaves a vortex sheet, which rapidly wraps into a pair of contrarotating vortices as shown in Fig. 1. The vortices appear to resist diffusion by small-scale turbulence but instead undergo a gradual large-scale instability propelled by their mutual induction. The instability was noted first by Scorer[1] and was analyzed in terms of mutual induction by Crow.[2] The essential predictions of the stability theory have been confirmed by flight tests,[3] and the theory has been extended to include the effects of core flow by Widnall and her co-workers at Massachusetts Institute of Technology (MIT).[4]

The mutual induction instability is the first in a sequence of processes that destroy the coherence and therefore the danger of a vortex wake. The subsequent processes, vortex linking and core bursting, fall outside the scope of linear stability theory but ordinarily occupy only a short interval toward the end of the wake lifespan. Figure 2 shows some photographic evidence obtained by Crow and Murman[5,6] during trailing vortex experiments at Moses Lake, Wash. The generating aircraft was the prototype Boeing 747 operating with 10° flaps, gear up, at a speed of 162 knots, and an altitude of 4900 ft. The aircraft passed through the picture field from left to right at $t=0$. The camera was aimed directly down the vortex trail at an angle from the horizontal of about 40°, and so the displacement waves are foreshortened by a factor of sin(40°) =0.64. The field shifts somewhat from frame to frame, because the camera was held by hand. The original photographs are in color and were taken at 10-sec intervals. The three reproduced in Fig. 2 were taken at 80, 100, and 120 sec after the passage of the 747. The instability is clearly visible at $t=80$ sec, but its amplitude still can be considered small. The instability is accelerating under nonlinear effects at $t=100$ sec, and the last vestige of coherence is disappearing only 20 sec later. Crow and Murman put those observations in quantitative form by plotting the maximum vortex separation $b_{\max}(t)$ minus the minimum separation $b_{\min}(t)$ divided by their sum

$$B(t)=(b_{\max}-b_{\min})/(b_{\max}+b_{\min})$$

$B(t)$ is zero in the absence of perturbation, grows exponentially with time during the linear instability, and becomes unity when the vortices link at points of minimum separation. Figure 3 is a plot of $B(t)$ in semilogarithmic coordinates, which display exponential growth as a straight line. The instability grows exponentially during the first 90 sec of vortex lifespan, then accelerates for about 20 sec, until the wake is too disorganized to permit a meaningful evaluation of B. Moore[7] drew much the same conclusion from a numerical study of the finite amplitude instability. He showed, in fact, that linear theory is surprisingly good right up to the time when the vortices link.

The evidence justifies extrapolating linear stability theory to predict wake lifespan. The mutual induction between trailing vortices, of course, determines only the slope of a growth curve of the kind shown in Fig. 3. Perturbations due to atmospheric turbulence or possibly to aircraft configuration set the height of the curve and therefore influence a prediction of wake lifespan. Atmospheric turbulence is treated as the source of perturbations in Secs. II-V of this paper, and active control surface manipulation is discussed in Sec. VI.

Atmospheric turbulence enters the stability theory in the form of inhomogeneous forcing terms. In principle the mean-square amplitude of vortex distortion can be calculated for any turbulent input spectrum, but here the calculations will be performed in detail under the assumption that eddies of size comparable to the instability wavelength lie in the Kolmogorov inertial subrange. The separation b between vortices behind a large jet aircraft is typically 100 ft, and the wavelength λ of the maximally amplified mode is 8.6 b,[2] or about 900 ft. The assumption that 900-ft eddies lie in the inertial subrange should be valid in the midst of the atmospheric boundary layer, where turbulence obeys Kolmogorov scaling at wavelengths as large as half a boundary-layer thickness.[8] The assumption is nevertheless highly restrictive and cannot apply to the important case of vortices generated near a runway. The conceptual advantage of Kolmogorov scaling is that

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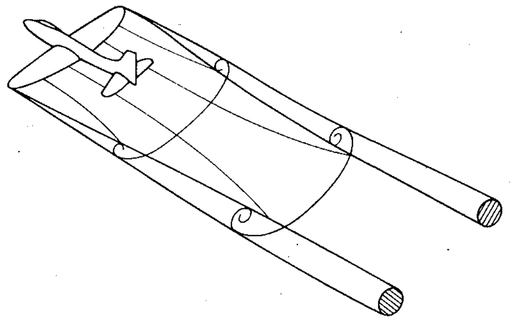

Fig. 1 Formation of aircraft trailing vortices.

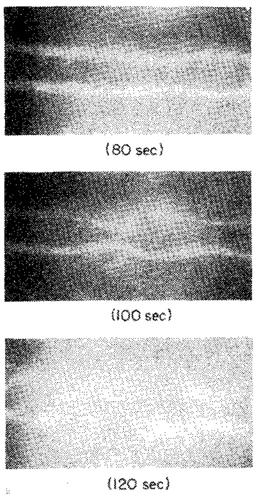

Fig. 2 Photographs of the mutual induction instability behind a Boeing 747 (from Ref. 5).

it permits the wake lifespan to be expressed as a universal function of a dimensionless turbulence intensity.

The turbulent energy spectrum takes the form $E(k) = \alpha\epsilon^{2/3} k^{-5/3}$ in the inertial subrange, where neither molecular viscosity nor the source of turbulent energy plays a role. α is a universal constant having a value around 1.5, ϵ is the rate of turbulent energy dissipation per unit mass, and k is the wave number $2\pi/\lambda$. The only physical parameter characterizing the inertial subrange is ϵ, with dimensions of $(\text{length})^2/(\text{time})^3$. Thus, the turbulent velocities causing deformations at the scale of the vortex separation b are of magnitude $v_\epsilon = (\epsilon b)^{1/3}$. The speed of descent of the vortex pair can be expressed in terms of vortex spacing b and circulation Γ: $v_\Gamma = \Gamma/2\pi b$. The ratio

$$\eta = v_\epsilon / v_\Gamma = (\epsilon b)^{1/3} / (\Gamma/2\pi b)$$

represents the intensity of the turbulence normalized on the strength of the vortex wake. The mean lifespan T of the vortex wake is a function of ϵ, b, Γ, and possibly also of a measure of vortex core structure, like diameter c. By dimensional analysis,

$$T(\epsilon, b\Gamma, c) = (2\pi b^2/\Gamma)\,\tau(\eta, c/b)$$

where $2\pi b^2/\Gamma$ is the time for the trailing vortices to move by mutual induction downward a distance b, and τ is a dimensionless function of η and c/b. The stability theory suggests only a weak dependence on c/b, which in any case should be nearly universal for aircraft having reasonable span loadings, and so

$$T = (2\pi b^2/\Gamma)\,\tau(\eta) \tag{1}$$

with sufficient generality. Kolmogorov scaling determines the basic structure of the lifespan prediction (1). The universal function $\tau(\eta)$ can be found from experiment or by a detailed analysis of the mechanisms of excitation and instability.

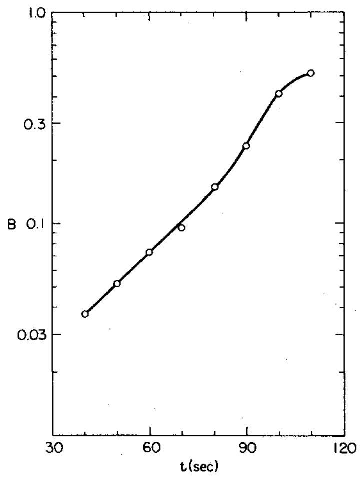

Fig. 3 Amplitude history of the instability shown in Fig. 2.

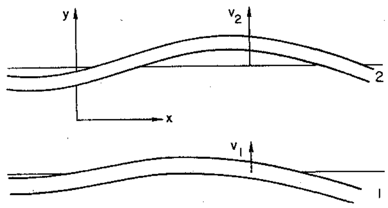

Fig. 4 Passive convection of trailing vortices in strong turbulence.

The present analysis involves three assumptions beyond Kolmogorov scaling:

1) The atmospheric turbulence is regarded as independent of the vortices. The velocity perturbation at a vortex core then can be decomposed into a vortex-induced component and a random component having Kolmogorov statistics. In reality, the aircraft vortices gradually will distort the ambient turbulence and cause departures from statistical homogeneity and isotropy. There seems to be no straightforward way to handle such departures, and the assumption that they are unimportant will have to be tested by comparing the calculated $\tau(\eta)$ to experiment.

2) The lifespan is determined by extrapolating linear theory to times when the displacement perturbations are comparable to the original vortex separation b. The actual lifespan is defined in terms of a mean-square displacement in Sec. II. The calculations of Ref. 7 and the evidence of Figs. 2 and 3 suggest that this assumption is entirely reasonable.

3) The atmospheric turbulence is assumed to be steady in coordinates moving downward with the vortices. At first glance, this assumption may appear untenable, but it is reasonable for eddies of scale 8.6 b. Wakes usually descend only a few separations b before losing their coherence. More will be said about this third assumption in Sec. III.

Relaxing any of the foregoing assumptions would change the precise form of $\tau(\eta)$ but not the fundamental similarity structure of Eq. (1). Currently the experimental data are too scattered to warrant a more refined analysis.

II. Bodily Convection in Strong Turbulence

It is instructive to begin the analysis by considering the case of strong turbulence, $\eta >> 1$. The results are of little practical importance, because the ambient turbulence would pose a greater danger than the trailing vortices. The case of strong turbulence however, illustrates some of the ideas involved in the more complicated study of Secs. III and IV, and also provides a useful asymptote for the general theory.

Figure 4 is a plan view of two vortices being distorted by atmospheric turbulence. The coordinate system moves downward with the vortices, with x pointing back along the aircraft flight path and y in the spanwise direction. The vortices deform almost passively in the strong turbulent field. If $v(x,y,z,t)$ is the y component of turbulent velocity, then the y components at the undisturbed locations of vortices 1 and 2 are as follows

$$v_1(x,t) = v(x, -b/2, 0, t)$$

$$v_2(x,t) = v(x, b/2, 0, t)$$

Suppose that $y_1(x,t)$ and $y_2(x,t)$ are the lateral displacements of the vortices from their undisturbed locations. As long as the displacements remain small

$$\partial v_1/\partial t = v_1, \quad \partial y_2/\partial t = v_2 \tag{2}$$

Only where y_2 is negative and y_1 is positive can the vortices touch and enter the terminal stages of link formation and core bursting. Thus, the quantity of interest is the difference displacement $y_S = y_2 - y_1$, representing divergence and convergence of the trailing vortices. The subscript S refers to the symmetric mode of instability introduced by Crow[2] and discussed further in Sec. III. If the turbulent field suffers no great change during the initial stages of growth, then Eq. (2) can be integrated to yield $y_S = (v_2 - v_1)t$, where v_2 and v_1 are to be evaluated at time zero.

At this stage the statistics of the turbulent flow come into play. The mean-square symmetric mode of displacement has the form

$$\langle y_S^2 \rangle = \langle (v_2 - v_1)^2 \rangle t^2 \tag{3}$$

where the angular brackets denote ensemble averages. If the vortex separation b lies in the inertial subrange, then the principle of local isotropy can be used to express the mean-square velocity difference in terms of energy dissipation ϵ and the Kolmogorov constant α

$$\langle (v_2 - v_1)^2 \rangle = (81/55)\Gamma(1/3)\alpha(\epsilon b)^{2/3} \tag{4}$$

The formula is taken from Ellison[9] and applies to the mean-square difference in the velocity component parallel to the separation between two points.

The final step is to define what is meant by vortex lifespan. It is reasonable to assume that the vortex danger will have subsided when the rms value of y_s is near b. For the purpose of this study, the lifespan T will be defined by the statement

$$\langle y_S^2 \rangle = b^2 \quad \text{when} \quad t = T \tag{5}$$

where y_S is obtained from linear theory analogous to Eq. (2). Criterion (5) is somewhat conservative when mutual induction causes the vortex displacements to be nearly sinusoidal. Thus if

$$y_1 = -(b/2)\cos kx, \quad y_2 = (b/2)\cos kx$$

then the vortices have just linked at points $kx = \pm\pi, \pm 3\pi$, and so forth, and yet the mean square of y_S has attained only the value

$$\langle y_S^2 \rangle = b^2 \langle \cos^2 kx \rangle = b^2/2$$

Criterion (5) nevertheless will be used throughout the range of turbulence intensities. Experiment can decide whether the T, so defined, accords with the cessation of vortex danger.

Equations (3-5) provide the vortex lifespan in strong turbulence

$$T = \left[\frac{55}{81\Gamma(1/3)\alpha}\right]^{1/2} \frac{b}{(\epsilon b)^{1/3}}$$

In universal dimensionless form,

$$\tau = 0.41/\eta \tag{6}$$

in line with Eq. (1) for $\alpha = 1.5$. Equation (6) is based on the assumption that $\eta >> 1$, but subsequent analysis will show that (6) is probably accurate whenever η is greater than 0.4.

III. Instability in Weak Turbulence

We now turn to the more usual case, where induction between the vortices is at least as important as convection by atmospheric turbulence. The coordinates for the more complete analysis are shown in Fig. 5. The dominant effect of mutual induction is the downward motion of the trailing vortices at a speed $\Gamma/2\pi b$. Displacement perturbations leave the circulation Γ intact but change the separation vector $\boldsymbol{R}_{ij}$ between elements of vortices i and j, where i and j each take on the values 1 or 2. The velocity change of a particular vortex element can be calculated from the Biot-Savart law as a sum of line integrals along the two vortices. The line integral representing induction of vortex i on itself diverges at $\boldsymbol{R}_{ii} = 0$ unless some account is taken of finite core diameter. Crow[2] removed the divergence by cutting off the self-induction integrals a distance d on either side of points where the velocity was being computed. The choice $d = 0.321c$ enabled the cutoff method to reproduce known exact solutions for vortex cores of diameter c in solid-body rotation. Moore and Saffman[10] showed that the cut off method is asymptotically valid for slender vortices, and they gave a general formula for the computation of d as a function of vorticity distribution.

The vortex displacements and turbulent velocities can be written conveniently as Fourier integrals with respect to their dependence on x. Thus

$$y_1(x,t) = \int_{-\infty}^{\infty} \hat{y}_1(k,t) e^{ikx} dk$$

$$v_1(x,t) = \int_{-\infty}^{\infty} \hat{v}_1(k,t) e^{ikx} dk$$

with similar formulas for the vertical displacement $z_1(x,t)$ and vertical component $w_1(x,t)$ of turbulent velocity at vortex 1. The corresponding Fourier integrals for vortex 2 differ only in the subscript. As long as the disturbances are small, Fourier modes with different wavenumbers are uncoupled, and each wavenumber k can be treated independently. This is the situation contemplated by Crow, and the equations for the evolution of the Fourier-transformed displacements $\hat{y}_1(k,t)$, $\hat{y}_2(k,t)$, and so forth, can be taken from Ref. 2

$$\partial\hat{y}_1/\partial t = (\Gamma/2\pi b^2)(-\hat{z}_1 + \psi\hat{z}_2 - \beta^2\omega\hat{z}_1) + \hat{v}_1 \tag{7a}$$

$$\partial\hat{y}_2/\partial t = (\Gamma/2\pi b^2)(\hat{z}_2 - \psi\hat{z}_1 + \beta^2\omega\hat{z}_2) + \hat{v}_2 \tag{7b}$$

$$\partial\hat{z}_1/\partial t = (\Gamma/2\pi b^2)(-\hat{y}_1 + \chi\hat{y}_2 + \beta^2\omega\hat{y}_1) + \hat{w}_1 \tag{7c}$$

$$\partial\hat{z}_2/\partial t = (\Gamma/2\pi b^2)(\hat{y}_2 - \chi\hat{y}_1 - \beta^2\omega\hat{y}_2) + \hat{w}_2 \tag{7d}$$

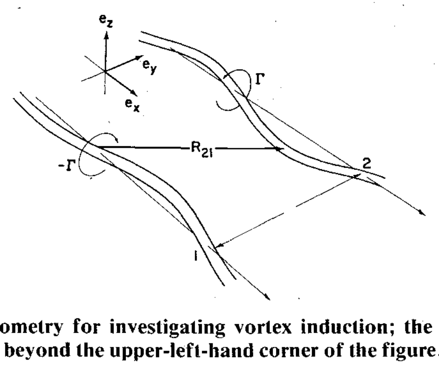

Fig. 5 Geometry for investigating vortex induction; the generating aircraft lies beyond the upper-left-hand corner of the figure.

The last terms on the right of Eqs. (7) represent turbulent convection. If they alone were present, then Eqs. (7) simply would be the Fourier-transformed versions of Eqs. (2). The terms in parentheses represent vortex induction, and they are multiplied by the inverse of the characteristic induction time $2\pi b^2/\Gamma$. The coupling coefficients ψ, χ, and ω are functions of a dimensionless vortex separation $\beta = \mathrm{kb}$ and cutoff distance $\delta = kd$. Thus,

$$\psi(\beta) = \beta^2 K_0(\beta) + \beta K_1(\beta)$$

$$\chi(\beta) = \beta K_1(\beta)$$

where K_0 and K_1 are modified Bessel functions of the second kind. Furthermore

$$\omega(\delta) = \tfrac{1}{2}\left[\frac{(\cos\delta - 1)}{\delta^2} + \frac{\sin\delta}{\delta} - Ci(\delta)\right]$$

where $Ci(\delta)$ is the integral cosine. The formulas and notation are those of Crow, who assigned distinct physical meanings to the three induction terms in brackets. The first term represents the displacements of one vortex diverging in the strain field of the other. The second term represents the velocity perturbations at the zeroth-order location of one vortex caused by the displacement perturbations of the other. The second term represents the velocity perturbations at the zeroth-order location of one vortex caused by the displacement perturbations of the other. The third term represents the tendency of vortex displacements to spin under self-induction. The first term always is destabilizing, and the third usually is stabilizing. Stability depends on the competition between divergence under mutual induction and spin under self-induction.

The vortex displacements can be decomposed into a mode that appears antisymmetric from the gound and a symmetric mode defined by the equations

$$\hat{y}_S = \hat{y}_2 - \hat{y}_1, \quad \hat{z}_S = \hat{z}_2 + \hat{z}_1$$

Equations for the evolution of the symmetric mode follow from (7)

$$\partial\hat{y}_S/\partial t = (\Gamma/2\pi b^2)(1 - \psi + \beta^2\omega)\hat{z}_S + (\hat{v}_2 - \hat{v}_1)$$

$$\partial\hat{z}_S/\partial t = (\Gamma/2\pi b^2)(1 + \chi - \beta^2\omega)\hat{y}_S + (\hat{w}_1 + \hat{w}_2)$$

The second equation can be combined with the time derivative of the first to yield an equation involving $\hat{y}_S$ alone. If $(\hat{v}_2 - \hat{v}_1)$ is small compared with $(\hat{w}_1 + \hat{w}_2)$ or changes little over times of order $2\pi b^2/\Gamma$, then the time derivative of $(\hat{v}_2 - \hat{v}_1)$ can be omitted, and the equation for $\hat{y}_S$ takes the form

$$(\partial^2\hat{y}_S/\partial t^2) - a^2\hat{y}_S = (a/\tan\theta_S)(\hat{w}_1 + \hat{w}_s) \qquad (8)$$

The following definitions have been introduced

$$a = (\Gamma/2\pi b^2)\left[(1 - \psi + \beta^2\omega)(1 + \chi - \beta^2\omega)\right]^{1/2}$$

$$\tan\theta_S = \left[\frac{(1 + \chi - \beta^2\omega)}{(1 - \psi + \beta^2\omega)}\right]^{1/2}$$

When a is real, it can be interpreted as the amplification rate of instability. The vortex displacements then diverge in planes fixed at the angle θ_S to the horizontal.

More should be said about the neglect of the time derivative $\partial(\hat{v}_2 - \hat{v}_1)/\partial t$ in Eq. (8). Even if the turbulence were "frozen" in atmosphere-fixed coordinates, the trailing vortices would seem to encounter changing velocities as they move downward. But the changes are slow at the wavenumbers that evoke maximum response from the vortices. The vortex wake can be regarded as a linear filter, with a divergent response around wavenumbers associated with the maximum of a^2. The response function a^2 has a maximum at a wavelength $\lambda = 8.6\ b$, and so the most effective eddies are an order of magnitude larger than the vortex separation itself. The geometry of the flow is shown in Fig. 6. The vortices spend a time of order $8.6\ (2\pi b^2/\Gamma)$ in each effective eddy, so the derivative $\partial(\hat{v}_2 - \hat{v}_1)/\partial t$ would be a factor of $(8.6)^{-1}$ smaller than the forcing term retained on the right of Eq. (8), even if $(\hat{v}_2 - \hat{v}_1)$ were of magnitude comparable to $\hat{w}_1 + \hat{w}_2)$. An eddy of scale 8.6 b is nearly coherent over a distance as short as b, so $(\hat{v}_2 - \hat{v}_1)$ will subtract to a small fraction of $\hat{v}_1$, whereas $(\hat{w}_1 + \hat{w}_2)$ will be close to $2\hat{w}_1 +$ On both counts the time derivative $\partial(\hat{v}_2 - \hat{v}_1)/\partial t$ can be dropped from the right of Eq. (8).

Equation (8) is to be solved subject to the initial conditions

$$\hat{y}_S = 0, \quad \partial\hat{y}_S/\partial t = \hat{v}_2 - \hat{v}_1 \text{ at } t = 0$$

The turbulent inputs $\hat{v}_1, \hat{v}_2, \hat{w}_1, \hat{w}_2$ are to be regarded as steady. The solution has the form

$$\hat{y}_S = Ae^{at} + Be^{-at} + C$$

with

$$A = [(\hat{w}_1 + \hat{w}_2)/2a\tan\theta_S] + [(\hat{v}_2 - \hat{v}_1)/2a]$$

$$B = [(\hat{w}_1 + \hat{w}_2)/2a\tan\theta_S] - [(\hat{v}_2 - \hat{v}_1)/2a]$$

$$C = -(\hat{w}_1 + \hat{w}_2)/a\tan\theta_S$$

The second terms in A and B are negligible to the degree of approximation inherent in Eq. (8). With comparable accuracy, the quantity $(\hat{w}_1 + \hat{w}_2)/2$ can be replaced with the vertical component of velocity Fourier-transformed along the x axis, which lies midway between the two vortices. Thus

$$(\hat{w}_1 + \hat{w}_2)/2 \approx \hat{w}$$

where

$$w(x,0,0) = \int_{-\infty}^{\infty} \hat{w}(k)e^{ikx}\mathrm{d}k$$

The self-consistent solution for the amplitude of the symmetric mode under strong mutual induction is as follows

$$\hat{y}_S = (2\hat{w}/a\tan\theta_S)[\cosh(at) - 1] \qquad (9)$$

Note that Eq. (9) does *not* reduce to the solution of Eqs. (2) in the limit of small a. When mutual induction is unimportant, $(\hat{v}_2 - \hat{v}_1)$ is the dominant forcing term and has to be retained throughout. When mutual induction is strong, the vertical component of velocity $\hat{w}$ is more important than the direct convection represented by $(\hat{v}_2 - \hat{v}_1)$.

If the turbulence is weak, then $\hat{y}_S$ becomes appreciable only when at is large. In that limit

$$\hat{y}_S = (\hat{w}/a\tan\theta_S)e^{at}$$

and the symmetric disturbance in physical space takes the form

$$y_S(x,t) = \int_{-\infty}^{\infty} \frac{\hat{w}(k)e^{a(k)t}}{a(k)\tan\theta_S(k)} e^{ikx}\mathrm{d}k \qquad (10)$$

It remains to study the statistics of (10) and thereby predict the mean wake lifespan.

IV. Mean-Square Amplitude

The mean-square value of y_S can be found by squaring both sides of Eq. (10) and taking an ensemble average

$$\langle y_S^2 \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\langle \hat{w}\hat{w}' \rangle e^{(a+a')t}}{aa' \tan\theta_S \tan\theta_S'} e^{i(k+k')x} \mathrm{d}k \mathrm{d}k' \tag{11}$$

The square on the right has been written as a double integral over the dummy variables k and k'. Unprimed quantities in the integrand are functions of k, and primed quantities are functions of k'. The Fourier amplitude $\hat{w}(k)$ is the only random function on the right of Eq. (10), so only the product $\langle w(k) w(k') \rangle$ is ensemble-averaged in the double integral. The integrand in Eq. (11) is valid wherever $a(k)t$ and $a(k')t$ are appreciable. A more accurate form of the integrand would have to be used where either $a(k)$ or $a(k')$ is zero, for example, but such regions contribute little to $\langle y_S^2 \rangle$ in the limit of large times.

The vertical component $w(x)$ of turbulent velocity is statistically homogeneous in x, and so the amplitude $\hat{w}(k)$ must be interpreted as a generalized Fourier transform. Homogeneity implies that the autocorrelation in wave-number space is proportional to the delta function of $(k+k')$ as explained by Tennekes and Lumley.[11] Thus

$$\langle \hat{w}(k) \hat{w}(k') \rangle = W(k) \delta(k+k')$$

where $W(k)$ is a well-behaved energy spectrum of the random process $w(k)$. The properties of the delta function can be used to perform the integral over k' in Eq. (11)

$$\langle y_S^2 \rangle = 2\int_0^{\infty} \frac{W(k)}{(a \tan\theta_S)^2} e^{2at} \mathrm{d}k \tag{12}$$

The amplification rate $a(k)$ and angle $\theta_S(k)$ do not depend on the sign of k, and so the integration is performed over positive k, with a factor of 2 to make up for negative values. Again, the integrand in (12) is accurate only where $a(k)t$ is appreciable and exp $[2a(k)t]$ is large, but other regions of wavenumber space contribute little to the mean-square deformation.

We now apply the method of steepest descents to evaluate the integral in (12). According to Fig. 9 of Ref. 2, the amplification rate $a(k)$ rises from zero at $k=0$, reaches a maximum at wavenumber k_m, and then decreases through zero and becomes imaginary. Only wavenumbers around k_m contribute to (12) when $\exp[2a(k_m)t]$ is large. The amplification rate can be expanded as a Taylor series for such wave numbers

$$a(k) = a(k_m) + a''(k_m)[(k-k_m)^2/2] + \ldots$$

where $a''(k_m)$ is negative, and the linear term is missing because $a(k_m)$ is a maximum. As time passes, the exponential factor in (12) becomes peaked sharply around k_m. The other factors can be evaluated at $k=k_m$ and placed in front of the integral. To dominant order

$$\langle y_S^2 \rangle = \frac{2W(k_m) e^{2a(k_m)t}}{[a(k_m) \tan\theta_S(k_m)]^2} \int_0^{\infty} e^{-|a''(k_m)|(k-k_m)^2 t} \mathrm{d}k$$

The integral now can be extended to $-\infty$ with no loss in accuracy and with the result that

$$\langle y_S^2 \rangle = \frac{2W(k_m) e^{2a(k_m)t}}{[a(k_m) \tan\theta_S(k_m)]^2} \left[\frac{\pi}{|a''(k_m)| t} \right]^{1/2} \tag{13}$$

For wavenumbers k_m in the Kolmogorov inertial subrange, the spectrum $W(k_m)$ has the universal form

$$W(k_m) = (12/55) \alpha \, \epsilon^{2/3} k_m^{-5/3}$$

according to Eq. (8.4.16) of Ref. 11. In conformity with the discussion of Sec. I and the treatment of Sec. II, we shall assume that Eq. (13) applies up to the vortex lifespan $t=T$, when $y_S^2 = b^2$.

The result is an implict equation for vortex lifespan in weak turbulence

$$b^2 = \frac{24}{55} \frac{\alpha \epsilon^{2/3} k_m^{-5/3} e^{2a(k_m)T}}{[a(k_m) \tan\theta_S(k_m)]^2} \left[\frac{\pi}{|a''(k_m)| T} \right]^{1/2} \tag{14}$$

The functions of k_m in Eq. (14) are available from the original stability theory. The functions depend weakly upon the ratio of cut off distance d to vortex separation b. For $d=0.321c$ and a realistic choice of core diameter c, Crow showed that

$$k_m = 0.73/b$$

$$a(k_m) = 0.83(\Gamma/2\pi b^2)$$

$$\tan\theta_S(k_m) = 1.11$$

He did not evaluate $a''(k_m)$, but it can be found from a parabolic fit to the amplification rate curve for the correct choice of d/b

$$|a''(k_m)| = 3.12(\Gamma/2\pi b^4)$$

If the Kolmogorov constant α is taken as 1.5, then the implicit lifespan formula (14) reduces to the form

$$\eta = 0.87\tau^{1/4} e^{-0.83\tau} \tag{15}$$

in the dimensionless variables of Eq. (1). Equation (15) determines the universal lifespan $\tau(\eta)$ for small values of the dimensionless turbulence intensity η and complements the opposite limit of Eq. (6).

V. Composite Lifespan

Equation (15) applies to the case of small η, and Eq. (6) to the case of large η. The assumptions underlying both asymptotic equations break down around $\eta \sim 1$, when mutual induction is important but not so dominant that the vortex displacements resemble sine waves of wavenumber k_m. Happily, however, the asymptotic analyses yield practically the same lifespans for a range of turbulence intensities centered around $\eta = 0.3$.

Figure 7 is a plot of Eqs. (6) and (15), both of which have been extended beyond their ostensible limits of validity. The coordinates are logarithmic, so Eq. (6) appears as a straight line of 45° slope. Equation (15) yields a double-valued function $\tau(\eta)$, and no lifespan estimate at all above $\eta = 0.5$. The lower branch of $\tau(\eta)$ is unphysical, however, because it corresponds to short lifespans inconsistent with the method of steepest descents. The most interesting feature of the curves in Fig. 7 is that they almost coincide in the interval $\eta = 0.22$ to 0.34. A uniformly valid plot of $\tau(\eta)$ hardly could depart much from the composite curve implied in Fig. 7.

Figure 8 shows the composite lifespan $\tau(\eta)$ obtained by fairing together the two curves in Fig. 7. For practical purposes, the function $\tau(\eta)$ of Fig. 8 can be taken as the universal lifespan, to the extent that the three assumptions listed at the end of Sec. I are valid. Also appearing on Fig. 8 are all of the relevant data known to be available at the time of writing. Each datum required a measurement of turbulence dissipation ϵ as well as knowledge of the vortex parameters Γ, b, and T. The data were provided by AeroVironment Inc., which specializes in flight tests and airborne turbulence measurements. The square data points pertaining to a Cessna 170 have been published before,[12] and the acquisition of the AeroCommander and Boeing 727 data is described by Tom-

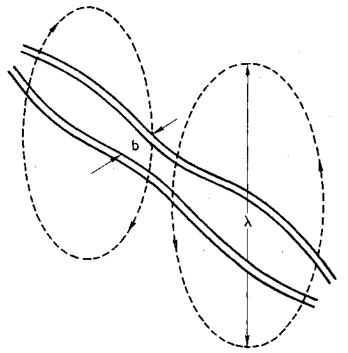

Fig. 6 Difference in scale between the excitatory eddies and the vortex separation; the vortices reside in a single eddy for times long compared with $2\pi b^2/\Gamma$.

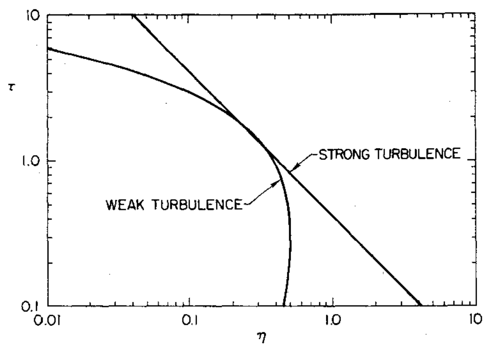

Fig. 7 Dimensionless wake lifespan according to the asymptotic formula for strong turbulence, Eq. (6), and for weak turbulence, Eq. (15).

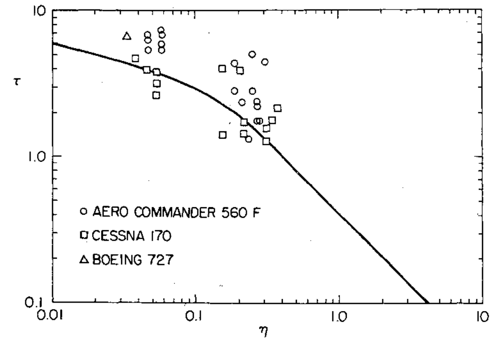

Fig. 8 Composite lifespan function compared to experimental data from Refs. 12 and 13.

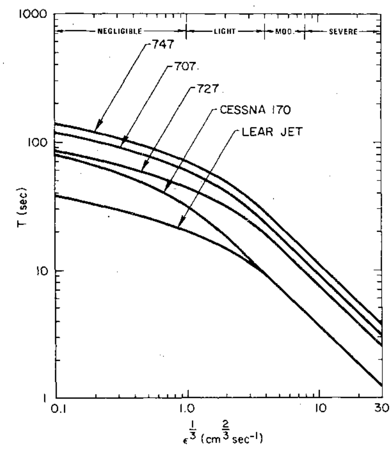

Fig. 9 Wake lifespans in physical units for various aircraft.

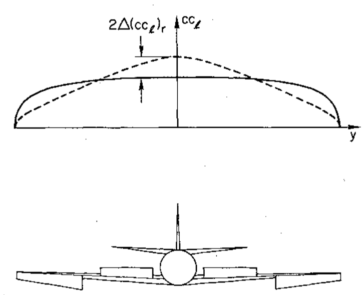

Fig. 10 Sloshing the lift distribution by symmetric oscillation of the lateral control surfaces.

bach and Bate.[13] There is considerable scatter, possibly because each point represents a single observation rather than an average, but more likely because the excitatory eddies were not wholly within the inertial subrange. Recall from Eq. (1) that τ must be *some* unique function of η, regardless of mechanical details, provided only that Kolmogorov scaling applies. Apart from the scatter, theory and experiment are in rough agreement, and we hope that the agreement will increase as more refined data become available.

Lifespan curves in dimensional units also are worth considering. Figure 9 is a collection of such curves for different aircraft. Each curve was obtained by combining the universal lifespan function of Fig. 8 with the vortex parameters Γ and b according to Eq. (1). The coordinates are again logarithmic, the ordinate representing lifespan T (sec), and the abscissa $\epsilon^{1/3}$ (cm$^{2/3}$/sec). The quantity $\epsilon^{1/3}$ correlates strongly with pilots' perception of ambient turbulence level, and ranges that pilots consider negligible, light, moderate, and severe are indicated on the plot. Except in the case of the Cessna 170, the turbulence must be moderate to severe before the strong-turbulence approximation (6) becomes applicable. The atmospheric turbulence is weak with respect to the wakes of the jet transports under normaly flying conditions, and Eq. (15) determines the lifespan. Notice how insensitive is the lifespan to turbulence intensity for negligible to light turbulence. The lifespan of the 747 wake increases from 70 to only 120 sec as the turbulence level $\epsilon^{1/3}$ drops from 1.0 to 0.1 cm$^{2/3}$/sec, corresponding to a thousand-fold decrease in ϵ itself. One can say that the lifespan of a 747 wake is about 100 sec without too much concern about the actual level of turbulence. A lifespan of 120 sec in weak turbulence, moreover, is consistent with the photographs of Fig. 2.

VI. Prospects of Artificial Disintegration

So far we have assumed that the aircraft itself is flying steadily, with no response to the surrounding turbulence and no changes in control-surface configuration. Any changes in flight path or lift distribution would have to be incorporated in the initial conditions for the solution of Eq. (8). Ordinarily such changes are probably negligible, but it is interesting to consider the advantage that might be gained by deliberately forcing the mutual induction instability.

In his beautifully photographed flight tests, Chevalier[3] forced the instability by propoising the aircraft. Porpoising is effective, but passenger comfort rules it out as a practical method for the abatement of vortex danger. Figure 10 illustrates a more acceptable method, suggested by Crow[14] and confirmed in the wind-tunnel tests of Bilanin and Widnall.[15] The aircraft controls are wired so that the outboard ailerons can oscillate symmetrically, whereas the inboard ailerons move in opposition to keep the total lift of the wing constant. The horizontal stabilizer trims any pitching tendency, and the net effect is to slosh the lift distribution in and out along the wingspan. The passengers feel only a slight longitudinal acceleration because of sinusoidal changes of induced drag, and even those are minimal if the unperturbed span loading is elliptical.

The plot in the upper half of Fig. 10 shows exaggerated extremes of the span load distribution cc_ℓ, where $c(y)$ is the wing chord and $c_\ell(y)$ is the local lift coefficient. The root loading $(cc_\ell)_r$ oscillates about its unperturbed value $(cc_\ell)_{r0}$ at an angular frequency Ω

$$(cc_\ell)_r = (cc_\ell)_{r0} + \Delta(cc_\ell)_r \sin\Omega t$$

leaving behind a symmetric vortex deformation in the horizontal plane

$$y_S = b\left[\frac{\Delta(cc_\ell)_r}{(cc_l)_{r0}}\right]\sin\left(\frac{\Omega x}{V}\right) \qquad (16)$$

Here V is the speed of the aircraft, and Eq. (16) can be regarded as the initial condition for the subsequent vortex instability. Equation (16) was derived in Ref. 14 under the assumption that spanwise vorticity shed behind the wing is negligible. A more refined analysis is presented in Ref. 15, but Eq. (16) is adequate for the present purposes.

The control-surface oscillations should be tuned to the most rapidly amplified mode of instability, which has a wave number k_m

$$\Omega = Vk_m = 0.73\ V/b$$

The speed of a 747 during approach to landing is about 240 fps and the vortex spacing is 109 ft, so that the appropriate forcing frequency is 1.60 rad/sec. The corresponding period is 3.91 sec, well beyond the range of structural modes of vibration. For the optimum choice of frequency, the initial condition (16) yields a lifespan

$$T = \frac{1}{a(k_m)} \log\left[\frac{2^{1/2}(cc_\ell)_{r0}}{\Delta(cc_\ell)_r}\right]$$

according to criterion (5), which is slightly different from the criterion advanced in Ref. 14. After reduction to dimensionless form

$$\tau = 1.20 \log\left[\frac{2^{1/2}(cc_\ell)_{r0}}{\Delta(cc_\ell)_r}\right]$$

If the oscillating part $\Delta(cc_\ell)_r$ of the root loading is only 5% of $(cc_\ell)_{r0}$, then $\tau = 4.0$. A 747 wake subjected to such an oscillation would last only 33 sec, a factor of 3 shorter than the lifespan observed by Crow and Murman. The 33-sec lifespan, moreover, would be guaranteed and not subject to the vagaries of atmospheric turbulence.

Fixed implements like tip spoilers can thicken vortex cores and reduce peak tangential velocities, but they can have little effect on the rolling moment imposed on a following aircraft. Forcing the mutual induction instability seems to be the only way to destroy the large-scale coherent flow responsible for vortex danger. The practical difficulty is that control cables in current aircraft constrain the lateral control surfaces to move antisymmetrically. Future jet transports are likely to have stability augmentation systems and independent, electronically actuated controls. A wake destruction mode then can be programed into the stability augmentation system, and the energy of the trailing vortices can be directed to their own destruction.

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